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$\bar\partial$-dressing and exact solutions for the (2 + 1)-dimensional Harry Dym equation

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Abstract. Exact implicit solutions with functional parameters of the (2 + 1)-dimensional Harry Dym (HD) equation are constructed by the $\bar\partial$-dressing method. The interrelation between the HD and modified Kadomtsev–Petviashvili (mKP) equations is established within the framework of the $\bar\partial$-dressing method. Knowledge of the mKP eigenfunctions allows the solutions of the (2 + 1)-dimensional HD equation to be represented in a simple parametric form.

1. Introduction

The Harry Dym (HD) equation $u_t + u^3u_{xxx} = 0$ is one of the most interesting and exotic soliton equations (see e.g. [1–6]). It was discovered in an unpublished paper by Harry Dym (see [1]) and rediscovered in more general form in [2] within the classical string problem. The $(1+1)$-dimensional HD equation possesses many properties which are typical of soliton equations (see e.g. [3, 4] and references therein). The inverse spectral transform method was applied to the HD equation in [5]. On the other hand, the HD equation has successfully resisted all attempts to construct its solutions in explicit form (see the review in [4]). The cusp solitons constructed in [5] are given by implicit formulae and can only be analysed numerically. The reciprocal link between the HD and KdV equations also provides implicit solutions since it includes a simultaneous change in both the dependent and independent variables [3, 4]. In the best case the necessity to invert the reciprocal transformation remains, so implicit solutions is a characteristic feature of the HD equation. The HD equation on the complex plane also arises in some physical problems such as string theory and the Saffman–Taylor and Hele–Shaw problems [2, 7–11].

The $(2+1)$-dimensional integrable generalization of the HD equation was proposed ten years ago in [12]. The whole infinite hierarchy of integrable equations (the 2DHD hierarchy) is associated with the 2DHD equation [13]. This hierarchy is the non-standard ($r = 2$) hierarchy within the Sato approach [13]. Similar to the $(1 + 1)$-dimensional case the 2DHD equation is connected to the modified Kadantsev–Petvashvili (mKP) and KP equations [14]. This connection is valid for all corresponding hierarchies [15]. In [15] some interesting reciprocal transformations were also presented.

In the present paper we consider the $(2 + 1)$-dimensional HD equation within the framework of the $\bar\partial$-dressing method. The $\bar\partial$-dressing method is a very strong and effective method to construct and simultaneously solve nonlinear partial differential equations [16–19] (see also [20]). In the HD case the $\bar\partial$-dressing method provides an infinite class of implicit exact solutions with functional parameters which correspond to generic degenerate $\bar\partial$ data.

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A comparison between the $\overline{\partial}$-dressing for the 2DHD and mKP equations gives rise to a simple interrelation between the corresponding wavefunctions. This interrelation provides a simple and convenient parametric form for the exact solutions of the 2DHD equation.

2. $\overline{\partial}$-dressing for the (2 + 1)-dimensional HD equation

The (2 + 1)-dimensional HD equation is of the form [12]

$$u_t + u^3 u_{xxx} + \frac{3}{u}\left(u^2 \partial_x^{-1}\left(\frac{u_y}{u^2}\right)\right)_y = 0. \tag{1}$$

It is equivalent to the compatibility condition for the linear system [12]

$$\begin{aligned} &\Psi_y + u^2 \Psi_{xx} = 0 \\ &\Psi_t + 4u^3 \Psi_{xxx} + 6u^2\left(u_x - \partial_x^{-1}\left(\frac{u_y}{u^2}\right)\right)\Psi_{xx} = 0. \end{aligned} \tag{2}$$

The $\overline{\partial}$-dressing method [16–20] starts with the non-local $\overline{\partial}$ problem

$$\frac{\partial \chi(\lambda,\overline{\lambda})}{\partial \overline{\lambda}} = \left(\hat{R} * \chi\right)(\lambda,\overline{\lambda}) = \int\int_C \mathrm{d}\lambda' \wedge \mathrm{d}\overline{\lambda'}\, \chi(\lambda',\overline{\lambda'}) R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda}) \tag{3}$$

where χ and R are scalar functions. It is assumed that the $\overline{\partial}$ equation is uniquely solvable and the function χ is normalized canonically, i.e. $\chi \to 1$ as $\lambda \to \infty$. The applicability of the $\overline{\partial}$-dressing method to the 2DHD equation (1) has been demonstrated in [20]. We will use the observation made in [20] to present here the complete $\overline{\partial}$-dressing scheme for the 2DHD equation.

First we introduce the dependence on the variables x, y, t into the $\overline{\partial}$ problem. It is fixed by the conditions

$$[D_x, \hat{R}] = [D_y, \hat{R}] = [D_t, \hat{R}] = 0 \tag{4}$$

where operators D_x, D_y, D_t are of the form

$$D_x = \partial_x + \frac{\mathrm{i}}{\lambda} f_x \qquad D_y = \partial_y + \frac{\mathrm{i}}{\lambda^2} + \frac{\mathrm{i}}{\lambda} f_y \qquad D_t = \partial_t + \frac{4\mathrm{i}}{\lambda^3} + \frac{\mathrm{i}}{\lambda} f_t \tag{5}$$

and $f(x, y, t)$ is a scalar real function. By virtue of (4) and (5) we have

$$R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};x,y,t) = R_0(\lambda',\overline{\lambda'};\lambda,\overline{\lambda}) \exp\left(F(\lambda',t) - F(\lambda,t)\right) \tag{6}$$

where R_0 is an arbitrary function and

$$F(\lambda,t) = \frac{\mathrm{i}}{\lambda} f + \frac{1}{\lambda^2} y + \frac{4\mathrm{i}}{\lambda^3} t. \tag{7}$$

According to the general approach (see e.g. [20]) one should construct the operators of the form

$$L = \sum_{n,l,m} u_{nlm}(x,y,t) D_x^n D_y^l D_t^m \tag{8}$$

which obey the following conditions

$$\left[\frac{\partial}{\partial \overline{\lambda}}, L\right]\chi = 0 \qquad \text{and} \qquad (L\chi)(\lambda) \to 0 \qquad \text{as } \lambda \to \infty. \tag{9}$$

For such operators L_i one has

$$L_i \chi = 0. \tag{10}$$

Linear equations (10) are the desired linear problems which give rise to nonlinear integrable systems [16–20].

It is not difficult to show that, in our case, one can construct the two independent operators which obey conditions (9). They are

$$\begin{aligned} L_1 &= D_y + u^2 D_x^2 \\ L_2 &= D_t + 4u^3 D_x^3 + 6u^2 \left(u_x - \partial_x^{-1}\left(\frac{u_y}{u^2}\right)\right) D_x^2 \end{aligned} \tag{11}$$

where

$$u = 1/f_x. \tag{12}$$

So linear problems (10) are of the form

$$D_y \chi + u^2 D_x^2 \chi = 0 \tag{13}$$

$$D_t \chi + 4u^3 D_x^3 \chi + 6u^2 \left(u_x - \partial_x^{-1}\left(\frac{u_y}{u^2}\right)\right) D_x^2 \chi = 0. \tag{14}$$

One obtains linear problems (2) in terms of the function Ψ defined by

$$\Psi = \chi \exp\left(\frac{\mathrm{i}}{\lambda} f + \frac{1}{\lambda^2} y + \frac{4\mathrm{i}}{\lambda^3} t\right). \tag{15}$$

By virtue of equations (13) and (14) the function $f(x, y, t)$ obeys the equations

$$f_y + \frac{f_{xx}}{f_x^2} + \frac{2\chi_{0x}}{\chi_0 f_x} = 0 \tag{16}$$

$$f_t - \tfrac{9}{2} f_y^2 + 3\partial_x^{-1}(f_x f_y)_y - \frac{2}{f_x^{3/2}}\left(\frac{1}{f_x^{1/2}}\right)_{xx} = 0 \tag{17}$$

where $\chi_0 := \chi(\lambda = 0, x, y, t)$. It is straightforward to check that if the function f obeys equation (17) then function $u = f_x^{-1}$ solves HD equation (1).

Function $f(x, y, t)$ and equations (16) and (17) play a fundamental role in the theory of the 2DHD equation.

3. Exact solutions

To construct the exact solutions of equation (17) and, hence, those of the 2DHD equation (1) one has to find the exact solutions of the $\bar{\partial}$ problem (3). As usual (see e.g. [20]) they correspond to the degenerate data R:

$$R_0(\lambda',\overline{\lambda'};\lambda,\overline{\lambda}) = \pi\sum_{k=1}^{N} f_k(\lambda',\overline{\lambda'})g_k(\lambda,\overline{\lambda}) \tag{18}$$

where f_k and g_k are arbitrary functions; these functions for real $u(x,y,t)$ (or equivalently for real $f(x,y,t)$) satisfy the conditions

$$\overline{f_k(\lambda,\overline{\lambda})} = f_k(-\overline{\lambda},-\lambda) \qquad \overline{g_k(\lambda,\overline{\lambda})} = g_k(-\overline{\lambda},-\lambda).$$

For such R_0 the integral equation which is equivalent to (3) can be solved explicitly. One obtains

$$\chi(\lambda,\overline{\lambda}) = 1 + \pi\sum_k h_k(f,y,t)\int\int_C \frac{\mathrm{d}\lambda'\wedge\mathrm{d}\overline{\lambda'}\,g_k(\lambda',\overline{\lambda'})\mathrm{e}^{-F(\lambda')}}{2\pi\mathrm{i}(\lambda'-\lambda)} \tag{19}$$

where

$$h_k(f,y,t) := \int\int_C \mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\,\chi(\lambda,\overline{\lambda})f_k(\lambda,\overline{\lambda})\mathrm{e}^{F(\lambda)}.$$

After some calculations one obtains

$$\chi_0 = 1 + \tfrac{1}{2}\sum_k q_k(f,y,t)(A^{-1})_{kl}p_l(f,y,t) = \det\left(\tilde{A}A^{-1}\right) \tag{20}$$

where the matrices A_{kl} and $\tilde{A}_{kl}$ have the form

$$A_{lk} = \delta_{lk} - \tfrac{1}{2}\partial_f^{-1}(q_k p_{l,f}) \qquad \tilde{A}_{kl} = \delta_{kl} + \tfrac{1}{2}\partial_f^{-1}(q_{l,f}p_k)$$

and

$$q_k := \int\int_C \frac{\mathrm{d}\lambda\wedge\overline{\lambda}}{\lambda}g_k(\lambda,\overline{\lambda})\mathrm{e}^{-F(\lambda)} \qquad p_k := -\mathrm{i}\int\int_C \mathrm{d}\lambda\wedge\mathrm{d}\overline{\lambda}\,f_k(\lambda,\overline{\lambda})\mathrm{e}^{F(\lambda)}.$$

In the last formulae $q_{l,f}$ and $p_{l,f}$ are partial derivatives of the function $f(x,y,t)$:

$$q_{l,f} := \frac{\partial q_l}{\partial f} \qquad p_{l,f} := \frac{\partial p_l}{\partial f}.$$

Due to the reality of $u(x,y,t)$ $\overline{q_l} = q_l$, $\overline{p_l} = p_l$ $(l = 1,\ldots,N)$.

So the exact solutions of equation (17) are presented by equation (16) where χ_0 is given by (20). This is a partial differential equation and, hence, the function f is defined implicitly. In the simplest case of one term in the sum (18) equation (16) looks like

$$f_y + \frac{f_{xx}}{f_x^2} + \frac{pq_{,f}}{1+\frac{1}{2}\partial_f^{-1}(pq_{,f})} + \frac{p_{,f}q}{1-\frac{1}{2}\partial_f^{-1}(p_{,f}q)} = 0. \tag{21}$$

The symbol ∂_f^{-1} in the above formulae by definition means the appropriately chosen inverse to the ∂_f integral operator: $\partial_f\partial_f^{-1} = 1$.

4. The interrelation between the HD and mKP equations

The construction of the exact solutions of the HD equation is simplified greatly if one uses the connection between the HD and mKP equations. This connection is a well known fact [14, 15]. Here we will describe it within the framework of the $\bar{\partial}$-dressing method. The mKP equation is of the form [12]

$$V_t + V_{\xi\xi\xi} - \tfrac{3}{2}V^2 V_\xi + 3\partial_\xi^{-1} V_{yy} - 3V_\xi \partial_\xi^{-1} V_y = 0. \tag{22}$$

It arises as the compatibility condition for the following linear problems [12]

$$\Psi_y^{\mathrm{mKP}} + \Psi_{\xi\xi}^{\mathrm{mKP}} + V\Psi_\xi^{\mathrm{mKP}} = 0 \tag{23}$$

$$\Psi_t^{\mathrm{mKP}} + 4\Psi_{\xi\xi\xi}^{\mathrm{mKP}} + 6V\Psi_{\xi\xi}^{\mathrm{mKP}} + (3V_\xi - 3\partial_\xi^{-1}V_y + \tfrac{3}{2}V^2)\Psi_\xi^{\mathrm{mKP}} = 0. \tag{24}$$

The mKP equation can be derived and solved by the $\bar{\partial}$-dressing method [21]. The corresponding ξ, y, t dependence of the $\bar{\partial}$ data R^{mKP} is of the form

$$R^{\mathrm{mKP}}(\lambda', \overline{\lambda'}; \lambda, \bar{\lambda}; \xi, y, t) = R_0^{\mathrm{mKP}}(\lambda', \overline{\lambda'}; \lambda, \bar{\lambda}) \exp(F^{\mathrm{mKP}}(\lambda') - F^{\mathrm{mKP}}(\lambda)) \tag{25}$$

where

$$F^{\mathrm{mKP}}(\lambda) = \frac{\mathrm{i}}{\lambda}\xi + \frac{1}{\lambda^2}y + \frac{4\mathrm{i}}{\lambda^3}t. \tag{26}$$

Let us now compare the $\bar{\partial}$ equations for the mKP and HD equations. It is not difficult to see that the mKP wavefunction $\chi^{\mathrm{mKP}}(\xi, y, t)$ after the change $\xi \to f(x, y, t)$ obeys the same $\bar{\partial}$ equation as the $\chi^{\mathrm{HD}}(x, y, t)$ (compare formulae (25)–(26) and (6)–(7)). Thus by virtue of the unique solvability of the $\bar{\partial}$ problems one obtains

$$\chi^{\mathrm{HD}}(x, y, t) = \chi^{\mathrm{mKP}}(f(x, y, t), y, t). \tag{27}$$

This equality presents the desired interrelation between the HD and mKP equations. So the transformation of the independent variables in this interrelation is given by

$$\xi = f(x, y, t) \qquad y \to y \qquad t \to t. \tag{28}$$

Relation (28) implies that

$$x = \Phi(\xi, y, t). \tag{29}$$

Using standard formulae for changes in independent and dependent variables (see e.g. [22, 4]), it is not difficult to show that, under the change (28) and (29), equations (16) and (17) are converted into the following

$$\Phi_y + \Phi_{\xi\xi} + V(\xi, y, t)\Phi_\xi = 0$$

$$\Phi_t + 4\Phi_{\xi\xi\xi} + 6V\Phi_{\xi\xi} + (3V_\xi - 3\partial_\xi^{-1}V_y + \tfrac{3}{2}V^2)\Phi_\xi = 0 \tag{30}$$

where

$$V(\xi, y, t) = -2\chi_{0\xi}/\chi_0 \tag{31}$$

which is just the mKP linear problems (23)–(24).

Thus function Φ in the change (29) is nothing other than mKP wavefunction $\Psi^{\mathrm{mKP}}(\xi, y, t)$. This fact is of major importance. Note that the connection between the mKP and HD equations via the change (29) where Φ is the mKP wavefunction was first established in [14] using a completely different approach. Using this observation one obtains from (12) the following representation for the solutions of the HD equation:

$$u(x, y, t) = \partial\Psi^{\mathrm{mKP}}/\partial\xi \tag{32}$$

$$x = \Psi^{\mathrm{mKP}}(\xi, y, t). \tag{33}$$

So for the known mKP wavefunction Ψ^{mKP} formulae (32) and (33) give a parametric form for the solutions of the HD equation. In most cases one cannot express ξ via x, y, t from (33) explicitly. Thus we have implicit solutions for the 2DHD equation (1).

Formulae (32) and (33) can be represented in different but equivalent forms. First, by substituting (28) into (33), we obtain

$$x = \Psi^{\mathrm{mKP}}(f(x, y, t), y, t). \tag{34}$$

Given Ψ^{mKP} it is the functional equation for the function $f(x, y, t)$. It is quite obvious that this functional equation is equivalent to differential equation (16). Hence, knowledge of the mKP wavefunctions allows us to reduce the problem of constructing the solutions of equation (16) and, consequently, of the 2DHD equation (1), to solving the purely functional equation (34).

Second, by substituting (33) into (32), one obtains

$$u(\Psi^{\mathrm{mKP}}(\xi, y, t), y, t) = \partial\Psi^{\mathrm{mKP}}/\partial\xi. \tag{35}$$

For a given solution u of the 2DHD equation it is the differential equation for the $\Psi^{\mathrm{mKP}}(\xi, y, t)$.

Both equations (34) and (35) can be used to solve the 2DHD equation. From a practical point of view the parametric representation seems more convenient. Note that the representation (32) and (33) has been considered within a different approach in [15].

5. Exact solutions via mKP wavefunctions

The mKP equation has been studied in detail in [21] (see also [20]). A number of different classes of solution have been constructed. Here we will only use the simplest one.

The most general exact solutions of the mKP equation correspond to the kernel R_0^{mKP} of the form

$$R_0^{\mathrm{mKP}}(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}) = \pi \sum_{k=1}^{N} f_k(\lambda', \overline{\lambda'}) g_k(\lambda, \overline{\lambda}) \tag{36}$$

where f_k and g_k are arbitrary functions; these functions for the real $V(\xi, y, t)$ satisfy the conditions [22]

$$\overline{f_k(\lambda, \overline{\lambda})} = f_k(-\overline{\lambda}, -\lambda), \overline{g_k(\lambda, \overline{\lambda})} = g_k(-\overline{\lambda}, -\lambda).$$

The corresponding wavefunction Ψ^{mKP} looks like [20, 21]

$$\Psi(\lambda) = \mathrm{e}^{F(\lambda)}\left(1 + \pi \sum_k h_k(\xi, y, t) \int\int_C \frac{\mathrm{d}\lambda' \wedge \mathrm{d}\overline{\lambda'}\, g_k(\lambda', \overline{\lambda'})\mathrm{e}^{-F(\lambda')}}{2\pi \mathrm{i}(\lambda' - \lambda)}\right) \tag{37}$$

where

$$h_k(\xi, y, t) := \int\int_C \mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}\, \chi(\lambda, \overline{\lambda}) f_k(\lambda, \overline{\lambda})\mathrm{e}^{F(\lambda)}$$

and

$$F(\lambda) = \frac{\mathrm{i}\xi}{\lambda} + \frac{y}{\lambda^2} + \frac{4\mathrm{i}t}{\lambda^3}.$$

Using (37) via (32) and (33), one obtains a very general class of exact solutions of the 2DHD equation.

Particular choices for the functions f_k and g_k give rise to particular specialized solutions of the mKP equation and, consequently, of the 2DHD equation.

First, let us choose

$$R_0^{\mathrm{mKP}}(\lambda', \lambda) = \frac{\pi}{2} \sum_{k=1}^{N} S_k \delta(\lambda' - \mathrm{i}\alpha_k)\delta(\lambda - \mathrm{i}\beta_k) \tag{38}$$

where α_k, β_k and S_k are arbitrary real constants and $\alpha_k \neq \beta_k$. The corresponding wavefunction $\Psi(\lambda)$ is of the form [20, 21]

$$\Psi(\lambda) = \mathrm{e}^{F(\lambda)}\left(1 + \sum_{k,m=1}^{N} \frac{S_k \exp(F(\mathrm{i}\alpha_k) - F(\mathrm{i}\beta_k)}{\beta_k + \mathrm{i}\lambda}(A^{-1})_{km}\right) \tag{39}$$

where

$$A_{mk} = \delta_{km} + \frac{S_k \exp(F(\mathrm{i}\alpha_k) - F(\mathrm{i}\beta_k))}{\alpha_m - \beta_k}. \tag{40}$$

For any pure imaginary λ such that $\lambda \neq \mathrm{i}\beta_k$ $(k = 1, \ldots, N)$ formula (39) provides, via (32) and (33), exact real solutions of the 2DHD equation. In the simplest case, $N = 1$, one has $(\alpha \equiv \alpha_1, \beta \equiv \beta_1)$

$$\begin{aligned} u(x, y, t) = {} & \frac{x}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - \frac{x}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \\ & \times \tanh\left[\frac{\xi}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) - \frac{y}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - 2t\left(\frac{1}{\alpha^3} - \frac{1}{\beta^3}\right) + \delta\right] \\ x = {} & \exp\left[\frac{\xi}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - \frac{y}{2}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) - 2t\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3}\right)\right] \Big/ \cosh\left[\frac{\xi}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\right. \\ & \left. - \frac{y}{2}\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) - 2t\left(\frac{1}{\alpha^3} - \frac{1}{\beta^3}\right) + \delta\right] \end{aligned} \tag{41}$$

where the parameters S, α and β are choosen so that

$$e^{2\delta} := S/(\alpha - \beta) > 0.$$

In particular if $\alpha = -\beta$ formulae (41) give

$$u(x, y, t) = -\frac{x}{\alpha}\left(1 - x^2 \exp\left(\frac{2y}{\alpha^2}\right)\right)^{1/2}. \tag{42}$$

It should be emphasized that formula (42) gives us the explicit (but stationary) solution of the 2DHD equation.

The $\overline{\partial}$ data R_0^{mKP} of the form (38) but with $\beta_k = \alpha_k (k = 1, \ldots, N)$ generate the plane rational solutions of the mKP equation [20, 21]. The corresponding wavefunction is

$$\Psi^{\mathrm{mKP}}(\lambda, \xi, y, t) = e^{F(\lambda)}\left(1 + \sum_{k,m=1}^{N} \frac{S_k}{\alpha_k + i\lambda}(A^{-1})_{km}\right) \tag{43}$$

where

$$A_{mk} = \left(\xi - \frac{2y}{\alpha_m} - \frac{12t}{\alpha_m^2} + \gamma_m\right)\delta_{km} - \frac{\alpha_m^2}{\alpha_m - \alpha_k}(1 - \delta_{km}). \tag{44}$$

So for any pure imaginary λ such that $\lambda \neq i\alpha_k$ $(k = 1, \ldots, N)$ expression (43) provides us, via (32) and (33), with the real exact solutions of the 2DHD equation. In particular, at $N = 1$ one has

$$u(x, y, t) = \left(\frac{x}{\alpha}\right) - x \Big/ \left[\xi - \frac{2y}{\alpha} - \frac{12t}{\alpha^2} + \frac{\alpha}{2}\right]$$

$$x = \exp\left(\frac{\xi}{\alpha} - \frac{y}{\alpha^2} - \frac{4t}{\alpha^3}\right) \Big/ \left[\xi - \frac{2y}{\alpha} - \frac{12t}{\alpha^2} + \frac{\alpha}{2}\right]. \tag{45}$$

Finally let us consider the kernel R_0^{mKP} which is more general than (38) and which corresponds to rational solutions of MKP equation. This kernel has the form [21]

$$R_0^{\mathrm{mKP}}(\lambda', \overline{\lambda'}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}\Big[S_k(\lambda', \lambda)\delta(\lambda' - \lambda_k)\delta(\lambda - \lambda_k)$$
$$+ \overline{S_k(-\overline{\lambda'}, -\overline{\lambda})\delta(\lambda' + \overline{\lambda_k})\delta(\lambda + \overline{\lambda_k})}\Big] \tag{46}$$

where λ_k are arbitrary complex constants and $S_k(\lambda', \lambda)$ are arbitrary complex functions. The corresponding real wavefunction of mKP equation in the simplest case of one term in the sum (46) may be choosen, for example, in the form

$$\Psi(\lambda; \xi, y, t) = e^{F(\lambda)}\chi(\lambda) + e^{F(-\overline{\lambda})}\chi(-\overline{\lambda}) \tag{47}$$

with

$$F(\lambda) = \frac{i\xi}{\lambda} + \frac{y}{\lambda^2} + \frac{4it}{\lambda^3} \qquad \text{and} \qquad \chi(\lambda) = 1 + \frac{iS}{\lambda_1 - \lambda}\chi(\lambda_1) - \frac{i\overline{S}}{\overline{\lambda_1} + \lambda}\chi(-\overline{\lambda_1})$$

where S and λ_1 are arbitrary complex parameters and

$$\chi(\lambda_1) = \overline{\chi(-\overline{\lambda_1})} = \frac{X_{\overline{\lambda_1}} + i\overline{\lambda_1}^2/(2\lambda_{1R})}{X_{\lambda_1}X_{\overline{\lambda_1}} - |\lambda_1|^4/(4\lambda_{1R}^2)}$$

$$X_{\lambda_1} = \xi - \frac{2iy}{\lambda_1} + \frac{12t}{\lambda_1^2} + \gamma_{\lambda_1} \qquad \lambda_1 = \lambda_{1R} + i\lambda_{1I}.$$

In the last formula $\gamma_{\lambda_1} = \overline{\gamma_{-\overline{\lambda_1}}}$ by definition.

Let us note that the choice of wavefunction Ψ^{mKP} in the form (47) is possible because the equation for the function $\chi = e^{-F(\lambda)}\Psi$ for the real $V(\xi, y, t)$ due to (23) admits the involution $\chi(-\overline{\lambda}) = \overline{\chi(\lambda)}$. For any complex λ such that $\lambda \neq \lambda_1$, $\lambda \neq -\overline{\lambda_1}$ formula (47) provides, via (32) and (33), the exact real solution of the 2DHD equation.

In a similar manner one can find the exact solutions of the 2DHD equation in parametrized form which are associated with breather-type and other solutions of the mKP equation [20, 21].

6. Conclusion

The results obtained in this paper can be easily extended to the whole 2DHD hierarchy. The 2DHD hierarchy can be constructed by $\overline{\partial}$-dressing via $\overline{\partial}$ problem (3) with $\overline{\partial}$ data (16) with

$$F^{\mathrm{hier}} = \sum_{k=1}^{\infty} \frac{1}{\lambda^k} x_k \tag{48}$$

where $x_1 = x$, $x_2 = y, \ldots$. Formula (16) is valid for the whole hierarchy together with the higher analogues of equation (17). Equation (21) is also true for all the hierarchy.

Finally the interrelation between the 2DHD hierarchy and the mKP hierarchy is given by the formulae (27)–(29). The analogue of the formulae (32) and (33) for the whole hierarchy looks like

$$u(x_1, \ldots, x_n) = \partial\Psi^{\mathrm{mKP}}/\partial\xi_1$$

$$x_1 = \Psi^{\mathrm{mKP}}(\xi_1, \xi_2, \ldots) \qquad x_k = \xi_k \qquad k = 2, 3, \ldots$$

where Ψ^{mKP} is the common wavefunction for the whole mKP hierarchy and $\xi_1, \xi_2, \ldots$ are independent variables for the mKP hierarchy.

In a similar manner one can analyse the (1 + 1)-dimensional HD equation and its connection with the mKdV equation within the framework of the $\overline{\partial}$-dressing method.

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